Bianchi-V Cosmological Solutions with Evolving Gravitational and Cosmological ‘Constants’

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Introduction

Bianchi Type-V Cosmology

The General Solution

• Model I: $\beta = 1$

Model II: The Emden-Fowler Approach

Numerical Solutions

• Case I: $w = 0$
• Case II: $w = 1/3$
• Case III: $w = -1$
The Bianchi Models

- Classes of non-standard cosmological models that are in principle spatially homogeneous but anisotropic
- Considered as a generalization of the well-known standard Friedman-Lemaître-Robertson-Walker (FLRW) models of cosmology
- Luigi Bianchi classified them according to their construction of homogeneous surfaces in space-time; constructed by the action of a 3-dimensional group of isometrics $G_3$ upon the space-like 3—surfaces
- Of great cosmological interest because they provide a way of studying the anisotropy at an early period of our universe’s expansion history
The Cosmological Constant $\Lambda$

- One of the most puzzling and unsolved problems in physics today is the so-called *Cosmological Constant Problem*
- Cosmology: regarded as a matter field with negative pressure (or as a vacuum energy density) that drives the accelerated expansion of the universe
- Value shows huge discrepancy with QFT-predicted value of vacuum energy
  - A new thought is required to explain this puzzle
  - Consider the cosmological models as varying vacuum energy density?
  - Chen\(^1\) considered $\Lambda$ proportional to $1/a^2$, and Carvalho et. al.\(^2\) studied the generalized form $\Lambda = \alpha/a^2 + \beta H^2$, which depends on adjustable parameters $\alpha$ and $\beta$ of the quantum field on a curved and expanding background, the Hubble parameter $H$ and the average scale factor of the universe $a(t)$

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\(^1\)Y.S. W. Chen (1990), PRD **41**, 695
\(^2\)J. C. Carvalho, J.A.S. Lima, and I. Waga (1992), PRD **46**, 2404
The Newtonian Constant $G$

- A coupling constant between the geometry of spacetime and energy in GR

- The universe evolves with time
  - Natural to assume that $G$ varies with time, too
  - First considered by Dirac $^3$
  - Many attempts to modify GR, but none of these efforts have yet been universally accepted
  - Recent interest in studying modifications of GR with variable cosmological and Newtonian ‘constants’

- Such studies include $^4$ $^5$ solving the EFEs for a Bianchi Type- $^\text{V}$ model with variable cosmological and Newtonian ‘constants’ for a stiff perfect fluid
  - The model has a singularity point, and $G$, $\Lambda$ and the shear parameter $\sigma$ decrease with cosmic time, with the model isotropizing at late times
  - The universe described by such a model expands at a constant rate ($i.e.$, the deceleration parameter equals zero)

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$^4$U. K. Dwivedi (2012), IJPMS 2 6
$^5$A. K. Yadav (2013), EJTP 10 28
General Solutions

 ► Existing solutions: special cases where the values of $\alpha$ and $\beta$ are chosen \textit{a priori}
  
  • The ‘cosmological constant’ decreases with time and it reaches a small positive value at late times

 ► In this work, we intend to show \textsuperscript{6,7}
  
  • The \textbf{exact general solution} of the EFEs for Bianchi type-$V$ models for a \textit{stiff perfect fluid} with variable $\Lambda$ and $G$ without making any constraints on the value of $\alpha$ and $\beta$ in the $\Lambda$ term, and to describe the behavior of the physical and kinematical parameters of the models
  
  • The \textbf{numerical solution} of the general system of the reducible EFEs of the Bianchi type$-V$ model with variable $G$ and $\Lambda$ for realistic perfect-fluid forms: \textit{baryonic matter, radiation} and \textit{dark energy}

\textsuperscript{6}Based on A. Alfedeeel, AA, M. Gubara (2018), Universe, 4(8), 83
\textsuperscript{7}A. Alfedeeel, AA, \textit{in preparation}
The Bianchi-V Cosmological Models

Assume the spatially homogeneous and anisotropic Bianchi type-V space-time

\[ ds^2 = -dt^2 + A^2(t)dx^2 + e^{2x} \left[ B^2(t)dy^2 + C^2(t)dz^2 \right] \]

with perfect-fluid matter forms with energy-momentum tensor

\[ T_{ij} = (p + \rho)u_i u_j + pg_{ij} \]

where \( \rho \) is matter density, \( u^i = \delta^i_1 = (-1, 0, 0, 0) \) is the normalized fluid four-velocity, which is a time-like quantity such that \( u^i u_i = -1 \), and \( p \) is the fluid's isotropic pressure. \( \rho \) and \( p \) are related through the barotropic equation of state

\[ p = w \rho \quad , \quad 0 \leq w \leq 1 \]

where \( w \) is the equation-of-state (EoS) parameter.

\(^8c = 1 \) is assumed throughout.
The EFEs with time-dependent $\Lambda$ and $G$ are given by

$$R_{ij} - \frac{1}{2}g_{ij}R = -8\pi G(t)T_{ij} + g_{ij}\Lambda(t)$$

where $R_{ij}$ and $g_{ij}$ are the Ricci and metric tensors respectively, and $R$ is the Ricci scalar. Explicitly, the EFEs for Bianchi-V read:

$$\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{A}\dot{C}}{AC} - \frac{3}{A^2} = 8\pi G(t)\rho + \Lambda(t)$$

$$\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} - \frac{1}{A^2} = -8\pi G(t)p + \Lambda(t)$$

$$\frac{\ddot{C}}{C} + \frac{\ddot{A}}{A} + \frac{\dot{C}\dot{A}}{CA} - \frac{1}{A^2} = -8\pi G(t)p + \Lambda(t)$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} - \frac{1}{A^2} = -8\pi G(t)p + \Lambda(t)$$

$$2\frac{\dot{A}}{A} - \frac{\dot{B}}{B} - \frac{\dot{C}}{C} = 0$$
The covariant divergence of the L.H.S of the EFEs produces

\[ 8\pi G \left[ \dot{\rho} + (\rho + p) \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) \right] + 8\pi \dot{G} + \dot{\Lambda} = 0 \]

while the conservation of the energy-momentum tensor yields

\[ \dot{\rho} + (\rho + p) \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) = 0 \]

Using these two equations together, we obtain

\[ \boxed{8\pi \rho \dot{G} + \dot{\Lambda} = 0} \]

This equation shows how \( \Lambda \) and \( G \) evolve with time and that they do not evolve independently of each other.
The average scale factor $a = a(t)$ for Bianchi-$V$ models is

$$a = (ABC)^{1/3}$$

and the generalized Hubble parameter $H$ is defined as

$$H = \frac{\dot{a}}{a} = \frac{1}{3} \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) = \frac{1}{3} (H_1 + H_2 + H_3)$$

where $H_1, H_2$ and $H_3$ are directional Hubble’s parameters along $x, y$ and $z$ directions respectively. The volume expansion $\theta$ is defined as

$$\theta = \nabla_i u^i = 3H$$

The deceleration parameter $q$ follows the usual definition

$$q = -\frac{\ddot{a}a}{\dot{a}^2} = -1 - \frac{\dot{H}}{H^2}$$
The shear module $\sigma$ parameter is given by

$$
\sigma^2 = \frac{1}{2} \sigma_{ij} \sigma^{ij} = \frac{1}{2} \left( \frac{\dot{A}^2}{A^2} + \frac{\dot{B}^2}{B^2} + \frac{\dot{C}^2}{C^2} \right) - \frac{\theta^2}{6}
$$

$$
= \frac{1}{3} \left( \frac{\dot{A}^2}{A^2} + \frac{\dot{B}^2}{B^2} + \frac{\dot{C}^2}{C^2} \right) - \frac{1}{3} \left( \frac{\dot{A} \dot{B}}{AB} + \frac{\dot{B} \dot{C}}{BC} + \frac{\dot{A} \dot{C}}{AC} \right)
$$

where the term $\sigma^{ij}$ represents the shear tensor. For this model, its scalar quantity

$$
\sigma = \frac{K}{a^3}
$$

where $K$ is a positive constant that is related to the anisotropy of the model. Having introduced these quantities, we can re-express the field equations in terms of $a$, $H$, $q$ and $\sigma$:

$$
\frac{\ddot{a}}{a} + 2 \frac{\dot{a}^2}{a^2} - \frac{2}{A^2} = 4\pi G(t) \rho (1 - w) + \Lambda(t)
$$

$$
8\pi G \rho - \Lambda = (2q - 1) H^2 - \sigma^2 + \frac{1}{A^2}
$$

$$
8\pi G \rho + \Lambda = 3H^2 - \sigma^2 - \frac{3}{A^2}
$$
Integrating
\[ \frac{\dot{A}}{A} - \frac{\dot{B}}{B} - \frac{\dot{C}}{C} = 0 \]
and absorbing the integration constant into \( A \) or \( B \), gives
\[ A^2 = BC \implies a(t) = A \]

Some more steps and we arrive at:
\[
\begin{align*}
\frac{B}{A} &= d_1 \exp \left( k_1 \int \frac{dt}{a^3} \right) \\
\frac{C}{B} &= d_2 \exp \left( k_2 \int \frac{dt}{a^3} \right) \\
\frac{C}{A} &= d_3 \exp \left( k_3 \int \frac{dt}{a^3} \right)
\end{align*}
\]

\( d_i, k_i \) (i=1,2,3) are all integration constants. These equations can be combined to give
\[
\begin{align*}
B &= m_1 a \exp \left( k \int \frac{dt}{a^3} \right) \\
C &= m_2 a \exp \left( -k \int \frac{dt}{a^3} \right)
\end{align*}
\]
where \( m_1, m_2 \) and \( k \) are constant values that depend on the undetermined constants \( d_1, d_2 \) and \( d_3 \).
Rewrite the acceleration equation assuming the parametrization of $\Lambda$

$$\Lambda(t) = \frac{\alpha}{a^2} + \beta H^2$$

so that

$$a\ddot{a} + (2 - \beta)\dot{a}^2 - (\alpha + 2) = 0$$

This is the generalized Friedman equation (GFE) of Bianchi type-V models. The first term $a\ddot{a}$ represents force per unit mass times distance or work of a system that is equivalent to a potential energy, the second term $(2 - \beta)\dot{a}^2$ is a kinetic energy and $(2 - \beta)$ is a kind of mass, and the last term is a kind of total energy of the system.
The General Solution

- For $\alpha \neq -2, \beta \neq 2$, the GFE is a non-linear second-order DE
- For the special cases of $\alpha = -2, \beta = 2$, the model has a scale factor solution that grows linearly with cosmic time, i.e.,
  \[ a(t) = C_1 t + a_0 \]
  where $C_1$ and $a_0$ are integration constants
- This solution has a constant expansion and an initially increasing $\Lambda$ that asymptotes to a constant value later on, and a decreasing $G$
- The model describes an expanding universe with an overall increasing volume, and that asymptotically approaches isotropy at late times
On the other hand, for $\beta \neq 2, 3$ and $\alpha = -2$, the model provides a solution of the form

$$a(t) = [(3 - \beta) (C_2 t + C_3)]^{\frac{1}{3-\beta}}$$

- This solution reduces to the linear expansion solution above when we fix $\beta = 2$
- Although the general behaviour of $V$ and $\sigma$ remains the same (infinitely increasing and asymptotically vanishing at time infinity, respectively), the behaviours of $\Lambda$ and $G$ depend on the choice of $\beta$
- For example, different behaviour is observed when $\beta = 1$ and $\beta = -0.5$ as shown
The non-linear \((\alpha \neq -2, \beta \neq 2)\) GFE can be integrated once to obtain

\[
\dot{a}^2 - \frac{c}{(\beta - 2)}a^{2(\beta-2)} = \left(\frac{\alpha + 2}{2 - \beta}\right)
\]

the solution of which is given by

\[
a_{2F1} \left(\frac{1}{2}, \frac{1}{2(\beta - 2)}, 1 + \frac{1}{2(\beta - 2)}; \frac{c}{(\alpha + 2)}a^{2(\beta-2)}\right) = \pm \sqrt{\frac{\alpha + 2}{2 - \beta}} \tau
\]

where \(\beta \neq 2\) and \(\alpha \neq -2\), \(\tau = t - t_0\) and \(2F1\) is the hypergeometric function.

- General form of the solution
- Cannot be simplified into an analytical expression for \(a\) as an explicit function of time unless we have to make some specific choices on the values of \(\alpha\) and \(\beta\)
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Model I: $\beta = 1$

If we choose $\beta = 1$, the GS reduces to a power solution for $a$:

$$a = \sqrt{\alpha + 2} \left[ \tau^2 + \frac{c}{(\alpha + 2)^2} \right]^{\frac{1}{2}}$$

$$H = \frac{\dot{a}}{a} = \frac{\tau}{\tau^2 + \frac{c}{(\alpha + 2)^2}}$$

$$\sigma = \frac{K}{a^3} = \frac{K}{\left[ \sqrt{\alpha + 2} \sqrt{\tau^2 + \frac{c}{(\alpha + 2)^2}} \right]^3}$$

$$\rho = \frac{\rho_0}{a^6} = \frac{\rho_0}{\left[ (\alpha + 2)\tau^2 + \frac{c}{\alpha + 2} \right]^3}$$

$$G = -\int \frac{\dot{\Lambda}}{8\pi \rho} dt = \frac{\alpha^3 + 5\alpha^2 + 8\alpha + 4}{8\pi \rho_0} \tau^4 - \frac{c}{4\pi \rho_0} \tau^2 + G_0$$

$$q = -\frac{\ddot{a}}{a^{\prime^2}} = -\frac{c}{(\alpha + 2)^2 \tau^2}$$
The three (anisotropic) scale factors can be computed to be:

\[ A = \sqrt{\alpha + 2} \sqrt{\tau^2 + \frac{c}{(\alpha + 2)^2}} \]

\[ B = B_0 \sqrt{\alpha + 2} \sqrt{\tau^2 + \frac{c}{(\alpha + 2)^2}} \exp \left[ \frac{k}{\sqrt{\alpha + 2}} \frac{\tau}{\sqrt{\tau^2 + \frac{c}{(\alpha + 2)^2}}} \right] \]

\[ C = C_0 \sqrt{\alpha + 2} \sqrt{\tau^2 + \frac{c}{(\alpha + 2)^2}} \exp \left[ -\frac{k}{\sqrt{\alpha + 2}} \frac{\tau}{\sqrt{\tau^2 + \frac{c}{(\alpha + 2)^2}}} \right] \]

where \( B_0 = m_1 e^{\text{const}} \), \( C_0 = m_2 e^{\text{const}} \) are constants of integration.
Initial point of singularity at $t = t_0$ and $c = 0$

As $t \to \infty$, $H$, $\theta$, $\sigma$, $\Lambda$, $q$, $\rho$ all decrease with time, whereas $G$, $V$, $A$, $B$ and $C$ increase.

Although the numerical values of $\Lambda$ and $q$ decrease asymptotically towards constant values, whether the model describes an accelerated or decelerated expansion solely depends on the sign of $c$.

- For example, a positive choice of $c$ describes an early accelerated expansion that eventually slows down to an asymptotically constant expansion, whereas a negative $c$ describes an early decelerated expansion that eventually asymptotes to a constant expansion at late times.

$|\sigma/\theta| \to 0$ as $t \to \infty$, thus indicating that the model approaches isotropy for large values of $t$.  

\footnote{As observed in Dwivedi (2012) IJPMS 26 as well.}
In redshift space, we can show that the deceleration parameter for the model can be given by

\[ q(z) = \frac{C(\alpha + 2)(1 + z)^2}{C(1 + z)^2 - (\alpha + 2)} \]

Transition from an early deceleration epoch (at large redshifts) to late-time acceleration (at small redshifts), with a mathematical singularity occurring at the value of \( z \) for which \( C(1 + z)^2 - (\alpha + 2) = 0 \), and hence the singular point can be shifted either way by choosing appropriate values of \( \alpha \) and \( C \)
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Comparing the non-linear GFE to the first integral of the Emden-Fowler equation

\[ \ddot{a} = r t^n a^m \]

which, for \( n = 0 \), is given by

\[ \dot{a}^2 - \frac{2r}{m+1} a^{m+1} = s \]

where \( m, n, r \) and \( s \) are constants, results in the particular solution

\[ a = \left[ \frac{2(m+1)}{r(m-1)^2} \right]^{\frac{1}{m-1}} t^{2-\frac{1}{1-m}} \]

Thus comparing the Emden-Fowler \((n = 0)\) equation and the GFE, we find that \( m = 2\beta - 5 \) and \( r = c \), and \( a(t) \) reads

\[ a = \left[ (\beta - 3) \sqrt{\frac{c}{\beta - 2}} \right]^{\frac{1}{3-\beta}} t \]

Generalization of all work that has been done on Bianchi-V cosmological models with varying \( \Lambda \) and \( G \)
Cosmological parameters of the model:

\[
H = \frac{1}{(3 - \beta)} \frac{1}{t}
\]

\[
\rho = \frac{\rho_0}{\left[ (\beta - 3) \sqrt{\frac{c}{\beta - 2}} t \right]^\frac{6}{3 - \beta}}
\]

\[
\sigma = \frac{K}{a^3} = \frac{K}{\left[ (\beta - 3) \sqrt{\frac{c}{\beta - 2}} t \right]^\frac{3}{3 - \beta}}
\]

\[
q = -\frac{\dddot{a}}{\dot{a}^2} = 2 - \beta
\]

Thus this model can have

- \( q > 0 \) (deceleration) if \( \beta < 2 \)
- \( q < 0 \) (acceleration) if \( \beta > 2 \)
- \( q = 0 \) (constant expansion) if \( \beta = 2 \)
The direct substitution of $a(t)$ in the $\Lambda$ expression yields:

$$\Lambda = \alpha \left[ (\beta - 3) \sqrt{\frac{c}{\beta - 2}} \, t \right]^{\frac{2}{\beta - 3}} + \frac{\beta}{(3 - \beta)^2} \frac{1}{t^2}$$

$$G = -\int \frac{\dot{\Lambda}}{8\pi \rho} \, dt = G_0 - \frac{c^2(\beta - 3)^4}{8\pi \rho_0 (\beta - 2)^3} \left[ C(\beta - 3) t^{\frac{2\beta}{3 - \beta}} - \frac{\alpha(\beta - 2)}{2} t^{\frac{4}{3 - \beta}} \right]$$

and the metric variables can be computed as

$$A = \left[ (\beta - 3) \sqrt{\frac{c}{\beta - 2}} \, t \right]^{\frac{1}{3 - \beta}}$$

$$B = B_0 \left[ (\beta - 3) \sqrt{\frac{c}{\beta - 2}} \, t \right]^{\frac{1}{3 - \beta}} \times \exp \left\{ \frac{k(\beta - 3)}{3} \left[ (\beta - 3) \sqrt{\frac{c}{\beta - 2}} \right]^{\frac{3}{\beta - 3}} t^{\frac{\beta}{3 - \beta}} \right\}$$

$$C = C_0 \left[ (\beta - 3) \sqrt{\frac{c}{\beta - 2}} \, t \right]^{\frac{1}{3 - \beta}} \times \exp \left\{ -\frac{k(\beta - 3)}{3} \left[ (\beta - 3) \sqrt{\frac{c}{\beta - 2}} \right]^{\frac{3}{\beta - 3}} t^{\frac{\beta}{3 - \beta}} \right\}$$
The above results show that the model has a point of initial singularity (Big Bang) at $t = 0$

The model represents a contracting universe solution where $V$ and both $\Lambda$ and $G$ simultaneously decrease, while $H$, $\theta$, $\sigma$ and $\rho$ all increase as time increases, with $H$ asymptotically approaching zero from below.

Note that the model produces a constant deceleration of the expansion for acceptable values of $\beta$.

The expansion rate $\theta$ of the universe slows down as time increases, with the expansion eventually stopping, the ratio of $|\sigma/\theta| \to \infty$ as $t \to \infty$ thus predicting that the universe in this model becomes more anisotropic at late times.

Moreover, as $t \to \infty$, the metric components $A$, $B$ and $C$ approach zero rapidly, and $a \to 0$, which, with an even more rapidly increasing energy density, potentially results in a Big Crunch.
Numerical Solutions

Here we use numerical integration methods and solve the coupled system:

\[
\begin{align*}
8\pi \rho \dot{G} + \dot{\Lambda} &= 0 \\
\frac{\ddot{a}}{a} + (2 - \beta) \frac{\dot{a}^2}{a^2} - \frac{2 + \alpha}{a^2} &= 4\pi G(t) \rho (1 - w)
\end{align*}
\]

for realistic perfect fluids dust matter, radiation and dark energy). Rewerite system as:

\[
\begin{align*}
\frac{da}{dt} &= Z \\
\frac{dZ}{dt} &= -\frac{a_1}{a} Z^2 + \frac{b_1}{a} + \frac{c_1 G}{a^{3w+2}} \\
\frac{dG}{dt} &= a_2 Z a^{3w} + b_2 Z^3 a^{3w} - c_2 \frac{Z}{a} G
\end{align*}
\]

where \( a_1 = (2 - \beta), \ b_1 = (2 + \alpha), \ c_1 = 4(1 - w)\rho_0, \ a_2 = \frac{\alpha(1 - \beta) - 2\beta}{4\rho_0}, \ b_2 = \frac{\beta(3 - \beta)}{4\rho_0}, \) and \( c_2 = \beta (1 - w) \)

It is important to know that the average scale factor of Bianchi type-V model \( a(t) \neq 0 \) as \( t \to 0 \) which implies \( \dot{a} \neq 0, \ \dot{Z} \neq 0 \) and \( \dot{G} \neq 0 \) are finite at the time of the Big Bang.
System numerically solved for non-stiff perfect fluid models

First attempt at numerical results for $\beta = 0.5$ and several values of $\alpha$ in the range $1 \leq \alpha \leq 2$

Preliminary observations:

- In each model, $V$ and $G$ increase with time except $G$ in the dark-energy-dominated model, which increases at early times and then decreases later on
- $\sigma$, $\sigma/\theta$ and $\Lambda$ decrease with time, which indicates that the universe in each model becomes isotropic at late times
- The deceleration parameter $q$ behaves differently depending on the EoS of each model, that is, for a matter-dominated universe where $w = 0$, it asymptotically increases from a negative value to reach a constant value at zero and for a radiation-dominated universe, it asymptotically increases from a negative to positive value less than 1 as $\alpha$ increases, and for a dark-energy-dominated universe when EoS $w = -1$, $q$ asymptotically negatively increases to approach a negative value less than $-1$
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The Case of Dust
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The Case of Radiation
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The Case of Dark Energy
Summary

General exact solutions for Bianchi Type-V cosmological models for a stiff perfect fluid with time varying cosmological and gravitational ‘constants’ without prior choice of the quantum-field-theoretically adjustable parameters $\alpha$ and $\beta$ that define the cosmological constant as $\Lambda = \alpha/a^2 + \beta H^2$

Two cosmological models obtained with a choice of suitably fixed values of $\beta$ and through the transformation of the generalized Friedman equation into a special case of the Emden-Fowler equation

The dynamical and kinematical parameters of each model are computed exactly
  - While one of these models results in a universe that asymptotically isotropizes at late times, the other becomes increasingly anisotropic

Simple numerical computation implemented to see the general behaviour of the cosmological parameters for more realistic perfect-fluid models; work in progress...

As more precise data become available, it will, in principle, be possible to constrain the different model parameters that we chose arbitrarily in this study to get a better picture of this class of cosmological models