Biianchi-V Cosmological Solutions with Evolving Gravitational and Cosmological 'Constants'

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The Bianchi Models

- Classes of non-standard cosmological models that are in principle spatially homogeneous but anisotropic
- Considered as a generalization of the well-known standard Friedman-Lemaître-Robertson-Walker (FLRW) models of cosmology
- Luigi Bianchi classified them according to their construction of homogeneous surfaces in space-time; constructed by the action of a 3-dimensional group of isometrics G_3 upon the space-like $3-$surfaces
- Of great cosmological interest because they provide a way of studying the anisotropy at an early period of our universe's expansion history

The Cosmological Constant Λ

- One of the most puzzling and unsolved problems in physics today is the so-called *Cosmological Constant Problem*
- Cosmology: regarded as a matter field with negative pressure (or as a vacuum energy density) that drives the accelerated expansion of the universe
- Value shows huge discrepancy with QFT-predicted value of vacuum energy
    - A new thought is required to explain this puzzle
    - Consider the cosmological models as varying vacuum energy density?
    - Chen [1] considered Λ proportional to $1/a^2$, and Carvalho et. al. [2] studied the generalized form $\Lambda = \alpha/a^2 + \beta H^2$, which depends on adjustable parameters α and β of the quantum field on a curved and expanding background, the Hubble parameter H and the average scale factor of the universe $a(t)$

[1] Y.S. W. Chen (1990), PRD **41**, 695

[2] J. C. Carvalho, J.A.S. Lima, and I. Waga (1992), PRD **46**, 2404

The Newtonian Constant G

- A coupling constant between the geometry of spacetime and energy in GR
- The universe evolves with time
  - Natural to assume that G varies with time, too
  - First considered by Dirac [3]
  - Many attempts to modify GR, but none of these efforts have yet been universally accepted
  - Recent interest in studying modifications of GR with variable cosmological and Newtonian ‘constants’
- Such studies include [4] [5]solving the EFEs for a Bianchi Type-V model with variable cosmological and Newtonian ‘constants’ for a stiff perfect fluid
  - The model has a singularity point, and G , Λ and the shear parameter σ decrease with cosmic time, with the model isotropizing at late times
  - The universe described by such a model expands at a constant rate (*i.e.*, the deceleration parameter equals zero)

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[3] P. A. M. Dirac (1937), “The Cosmological Constants", Nature **139**, 323

[4] U. K. Dwivedi (2012), IJPMS **2** 6

[5] A. K. Yadav (2013), EJTP **10** 28

General Solutions

- Existing solutions: special cases where the values of α and β are chosen *a priori*
  - The 'cosmological constant' decreases with time and it reaches a small positive value at late times
- In this work, we intend to show [6] [7]
  - The exact general solution of the EFEs for Bianchi type-V models for a stiff perfect fluid with variable Λ and G without making any constraints on the value of α and β in the Λ term, and to describe the behavior of the physical and kinematical parameters of the models
  - The numerical solution of the general system of the reducible EFEs of the Bianchi type$-V$ model with variable G and Λ for realistic perfect-fluid forms: baryonic matter, radiation and dark energy

[6] Based on A. Alfedeel, AA, M. Gubara (2018), Universe, **4**(8), 83

[7] A. Alfedeel, AA, *in preparation*

The Bianchi-V Cosmological Models

Assume the spatially homogeneous and anisotropic Bianchi type-V space-time [8]

$$\boxed{ds^2 = -dt^2 + A^2(t)dx^2 + e^{2x}\left[B^2(t)dy^2 + C^2(t)dz^2\right]}$$

with perfect-fluid matter forms with energy-momentum tensor

$$T_{ij} = (p+\rho)u_i u_j + p g_{ij}$$

where ρ is matter density, $u^i = \delta^i_t = (-1, 0, 0, 0)$ is the normalized fluid four-velocity, which is a time-like quantity such that $u^i u_i = -1$, and p is the fluid's isotropic pressure. ρ and p are related through the barotropic equation of state

$$p = w\rho \ , \qquad 0 \leq w \leq 1$$

where w is the equation-of-state (EoS) parameter.

[8] $c = 1$ is assumed throughtout.

The EFEs with time-dependent Λ and G are given by

$$R_{ij} - \frac{1}{2} g_{ij} R = -8\pi G(t) T_{ij} + g_{ij}\Lambda(t)$$

where R_{ij} and g_{ij} are the Ricci and metric tensors respectively, and R is the Ricci scalar. Explicitly, the EFEs for Bianchi-V read:

$$\begin{aligned}
&\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{A}\dot{C}}{AC} - \frac{3}{A^2} = 8\pi G(t)\rho + \Lambda(t) \\
&\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} - \frac{1}{A^2} = -8\pi G(t)p + \Lambda(t) \\
&\frac{\ddot{C}}{C} + \frac{\ddot{A}}{A} + \frac{\dot{C}\dot{A}}{CA} - \frac{1}{A^2} = -8\pi G(t)p + \Lambda(t) \\
&\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} - \frac{1}{A^2} = -8\pi G(t)p + \Lambda(t) \\
&2\frac{\dot{A}}{A} - \frac{\dot{B}}{B} - \frac{\dot{C}}{C} = 0
\end{aligned}$$

The covariant divergence of the L.H.S of the EFEs produces

$$8\pi G\left[\dot{\rho}+(\rho+p)\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right)\right]+8\pi\dot{G}+\dot{\Lambda}=0$$

while the conservation of the energy-momentum tensor yields

$$\dot{\rho}+(\rho+p)\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right)=0$$

Using these two equations together, we obtain

$$\boxed{8\pi\rho\dot{G}+\dot{\Lambda}=0}$$

This equation shows how Λ and G evolve with time and that they do not evolve independently of each other.

The average scale factor $a = a(t)$ for Bianchi-V models is

$$a = (ABC)^{1/3}$$

and the generalized Hubble parameter H is defined as

$$H = \frac{\dot{a}}{a} = \frac{1}{3}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) = \frac{1}{3}(H_1 + H_2 + H_3)$$

where H_1, H_2 and H_3 are directional Hubble's parameters along x, y and z directions respectively. The volume expansion θ is defined as

$$\theta = \nabla_i u^i = 3H$$

The deceleration parameter q follows the usual definition

$$q = -\frac{\ddot{a}a}{\dot{a}^2} = -1 - \frac{\dot{H}}{H^2}$$

The shear module σ parameter is given by

$$\sigma^2 = \frac{1}{2}\sigma_{ij}\sigma^{ij} = \frac{1}{2}\left(\frac{\dot{A}^2}{A^2} + \frac{\dot{B}^2}{B^2} + \frac{\dot{C}^2}{C^2}\right) - \frac{\theta^2}{6}$$

$$= \frac{1}{3}\left(\frac{\dot{A}^2}{A^2} + \frac{\dot{B}^2}{B^2} + \frac{\dot{C}^2}{C^2}\right) - \frac{1}{3}\left(\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{A}\dot{C}}{AC}\right)$$

where the term σ^{ij} represents the shear tensor. For this model, its scalar quantity

$$\sigma = \frac{K}{a^3}$$

where K is a positive constant that is related to the anisotropy of the model. Having introduced these quantities, we can re-express the field equations in terms of a, H, q and σ:

$$\begin{aligned}
&\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} - \frac{2}{A^2} = 4\pi G(t)\rho(1-w) + \Lambda(t) \\
&8\pi Gp - \Lambda = (2q-1)H^2 - \sigma^2 + \frac{1}{A^2} \\
&8\pi G\rho + \Lambda = 3H^2 - \sigma^2 - \frac{3}{A^2}
\end{aligned}$$

Integrating

$$2\frac{\dot{A}}{A} - \frac{\dot{B}}{B} - \frac{\dot{C}}{C} = 0$$

and absorbing the integration constant into A or B, gives

$$A^2 = BC \implies a(t) = A$$

Some more steps and we arrive at:

$$\begin{aligned} \frac{B}{A} &= d_1 \exp\left(k_1 \int \tfrac{dt}{a^3}\right) \\ \frac{C}{B} &= d_2 \exp\left(k_2 \int \tfrac{dt}{a^3}\right) \\ \frac{C}{A} &= d_3 \exp\left(k_3 \int \tfrac{dt}{a^3}\right) \end{aligned}$$

d_i , k_i (i=1,2,3) are all integration constants. These equations can be combined to give

$$\begin{aligned} B &= m_1 \; a \; \exp\left(k \int \tfrac{dt}{a^3}\right) \\ C &= m_2 \; a \; \exp\left(-k \int \tfrac{dt}{a^3}\right) \end{aligned}$$

where m_1 , m_2 and k are constant values that depend on the undetermined constants d_1, d_2 and d_3.

Rewrite the acceleration equation assuming the parametrization of Λ

$$\Lambda(t) = \frac{\alpha}{a^2} + \beta H^2$$

so that

$$\boxed{a\ddot{a} + (2-\beta)\dot{a}^2 - (\alpha+2) = 0}$$

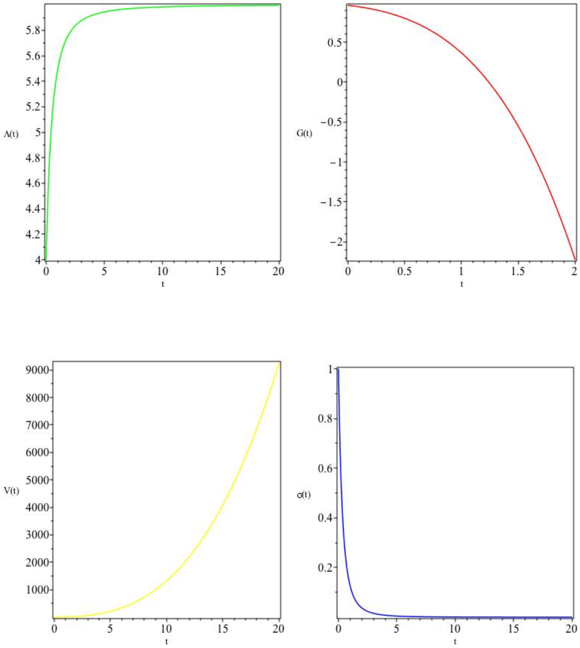
This is the generalized Friedman equation (GFE) of Bianchi type-V models. The first term $a\ddot{a}$ represents force per unit mass times distance or work of a system that is equivalent to a potential energy, the second term $(2-\beta)\dot{a}^2$ is a kinetic energy and $(2-\beta)$ is a kind of mass, and the last term is a kind of total energy of the system.

The General Solution

- For $\alpha \neq -2, \beta \neq 2$, the GFE is a non-linear second-order DE
- For the special cases of $\alpha = -2, \beta = 2$, the model has a scale factor solution that grows linearly with cosmic time, *i.e.*,

$$a(t) = C_1 t + a_0$$

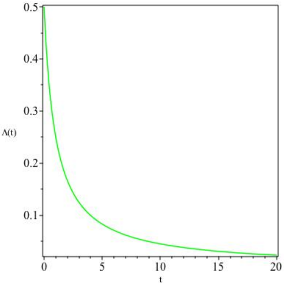
  where C_1 and a_0 are integration constants
- This solution has a constant expansion and an initially increasing Λ that asymptotes to a constant value later on, and a decreasing G
- The model describes an expanding universe with an overall increasing volume, and that asymptotically approaches isotropy at late times

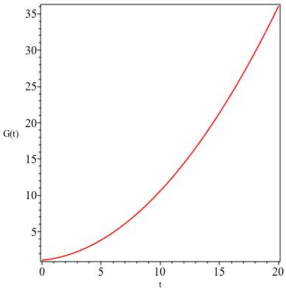

Λ(t)
G(t)
V(t)
σ(t)
t

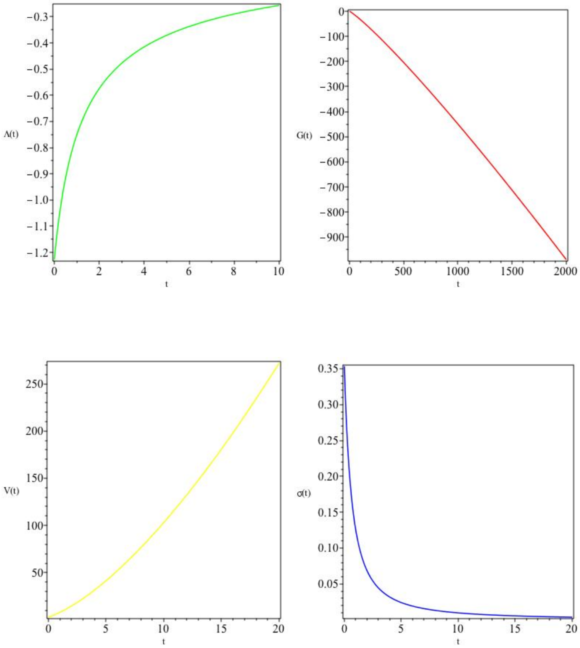
On the other hand, for $\beta \neq 2\,,3$ and $\alpha = -2$, the model provides a solution of the form

$$a(t) = \left[(3-\beta)\left(C_2 t + C_3\right)\right]^{\frac{1}{3-\beta}}$$

- This solution reduces to the linear expansion solution above when we fix $\beta = 2$
- Although the general behaviour of V and σ remains the same (infinitely increasing and asymptotically vanishing at time infinity, respectively), the behaviours of Λ and G depend on the choice of β
- For example, different behaviour is observed when $\beta = 1$ and $\beta = -0.5$ as shown






Λ(t)
t
G(t)
t
V(t)
t
σ(t)
t

The non-linear ($\alpha \neq -2$, $\beta \neq 2$) GFE can be integrated once to obtain

$$\dot{a}^2 - \frac{c}{(\beta - 2)} \; a^{2(\beta-2)} = \left(\frac{\alpha + 2}{2 - \beta}\right)$$

the solution of which is given by

$$a \; {}_2F_1 \left(\frac{1}{2}, \frac{1}{2(\beta-2)}, 1 + \frac{1}{2(\beta-2)}; \frac{c}{(\alpha+2)} a^{2(\beta-2)}\right) = \pm\sqrt{\frac{\alpha+2}{2-\beta}}\tau$$

where $\beta \neq 2$ and $\alpha \neq -2$, $\tau = t - t_0$ and ${}_2F_1$ is the hypergeometric function.

- General form of the solution
- Cannot be simplified into an analytical expression for a as an explicit function of time unless we have to make some specific choices on the values of α and β

Model I: $\beta = 1$

If we choose $\beta = 1$, the GS reduces to a power solution for a:

$$a = \sqrt{\alpha+2}\left[\tau^2 + \frac{c}{(\alpha+2)^2}\right]^{\frac{1}{2}}$$

$$H = \frac{\dot{a}}{a} = \frac{\tau}{\tau^2 + \frac{c}{(\alpha+2)^2}}$$

$$\sigma = \frac{K}{a^3} = \frac{K}{\left[\sqrt{\alpha+2}\sqrt{\tau^2 + \frac{c}{(\alpha+2)^2}}\right]^3}$$

$$\rho = \frac{\rho_0}{a^6} = \frac{\rho_0}{\left[(\alpha+2)\tau^2 + \frac{c}{\alpha+2}\right]^3}$$

$$G = -\int \frac{\dot{\Lambda}}{8\pi\ \rho}dt = \frac{\alpha^3 + 5\alpha^2 + 8\alpha + 4}{8\pi\rho_0}\ \tau^4 - \frac{c}{4\pi\rho_0}\ \tau^2 + G_0$$

$$q = -\frac{\ddot{a}a}{\dot{a}^2} = -\frac{c}{(\alpha+2)^2\tau^2}$$

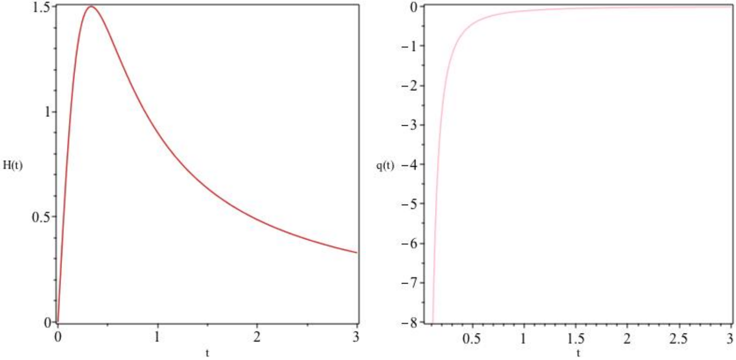
The three (anisotropic) scale factors can be computed to be:

$$A = \sqrt{\alpha + 2}\sqrt{\tau^2 + \frac{c}{(\alpha+2)^2}}$$

$$B = B_0\sqrt{\alpha + 2}\sqrt{\tau^2 + \frac{c}{(\alpha+2)^2}}\ \exp\left[\frac{k}{\sqrt{\alpha+2}}\ \frac{\tau}{\sqrt{\tau^2 + \frac{c}{(\alpha+2)^2}}}\right]$$

$$C = C_0\sqrt{\alpha + 2}\sqrt{\tau^2 + \frac{c}{(\alpha+2)^2}}\exp\left[-\frac{k}{\sqrt{\alpha+2}}\ \frac{\tau}{\sqrt{\tau^2 + \frac{c}{(\alpha+2)^2}}}\right]$$

where $B_0 = m_1 e^{\text{const}}$, $C_0 = m_2 e^{\text{const}}$ are constants of integration.



H(t)
t
q(t)
t

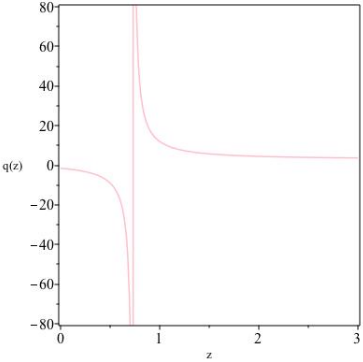
- Initial point of singularity at $t = t_0$ and $c = 0$
- As $t \to \infty$, H , θ , σ , Λ , q , ρ all decrease with time, whereas G, V, A , B and C increase
- Although the numerical values of Λ and q decrease asymptotically towards constant values, whether the model describes an accelerated or decelerated expansion solely depends on the sign of c
  - For example, a positive choice of c describes an early accelerated expansion that eventually slows down to an asymptotically constant expansion, whereas a negative c describes an early decelerated expansion that eventually asymptotes to a constant expansion at late times
- $|\sigma/\theta| \to 0$ as $t \to \infty$, thus indicating that the model approaches isotropy for large values of t [9]

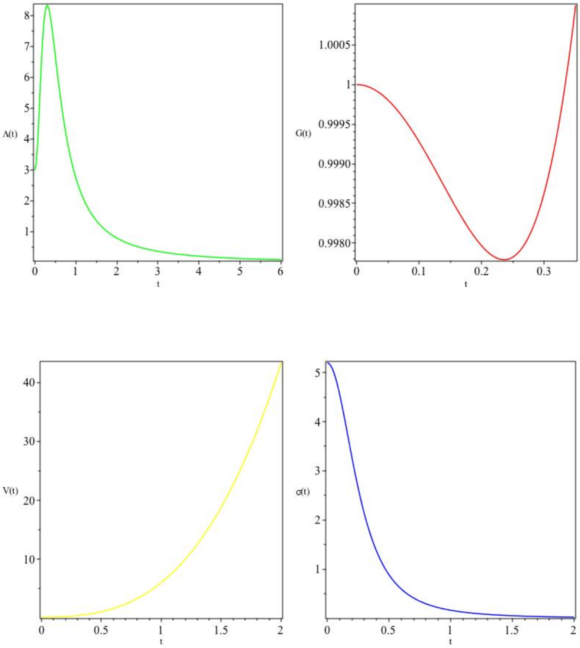
[9] As observed in Dwivedi (2012) IJPMS **2** 6 as well.

In redshift space, we can show that the deceleration parameter for the model can be given by

$$q(z) = \frac{C(\alpha+2)(1+z)^2}{C(1+z)^2 - (\alpha+2)}$$

- Transition from an early deceleration epoch (at large redshifts) to late-time acceleration (at small redshifts), with a mathematical singularity occuring at the value of z for which $C(1+z)^2 - (\alpha+2) = 0$, and hence the singular point can be shifted either way by choosing appropriate values of α and C





Λ(t)
t
G(t)
t
V(t)
t
σ(t)
t

Comparing the non-linear GFE to the first integral of the Emden-Fowler equation

$$\ddot{a} = rt^n \; a^m$$

which, for $n = 0$, is given by

$$\dot{a}^2 - \frac{2r}{m+1} \; a^{m+1} = s$$

where m , n , r and s are constants, results in the particular solution

$$\boxed{a = \left[\frac{2(m+1)}{r(m-1)^2} \right]^{\frac{1}{m-1}} \quad t^{\frac{2}{1-m}}}$$

Thus comparing the Emden-Fowler $(n = 0)$ equation and the GFE, we find that $m = 2\beta - 5$ and $r = c$, and $a(t)$ reads

$$a = \left[(\beta - 3) \sqrt{\frac{c}{\beta - 2}} \;\; t \right]^{\frac{1}{3-\beta}}$$

- Generalization of all work that has been done on Bianchi-V cosmological models with varying Λ and G

Cosmological parameters of the model:

$$H = \frac{1}{(3-\beta)}\ \frac{1}{t}$$

$$\rho = \frac{\rho_0}{\left[(\beta-3)\ \sqrt{\frac{c}{\beta-2}}\ \ t\right]^{\frac{6}{3-\beta}}}$$

$$\sigma = \frac{K}{a^3} = \frac{K}{\left[(\beta-3)\ \sqrt{\frac{c}{\beta-2}}\ \ t\right]^{\frac{3}{3-\beta}}}$$

$$q = -\frac{\ddot{a}\ a}{\dot{a}^2} = 2-\beta$$

Thus this model can have

- $q > 0$ (deceleration) if $\beta < 2$
- $q < 0$ (acceleration) if $\beta > 2$
- $q = 0$ (constant expansion) if $\beta = 2$

The direct substitution of $a(t)$ in the Λ expression yields:

$$\Lambda = \alpha \left[(\beta - 3) \sqrt{\frac{c}{\beta - 2}} \;\; t \right]^{\frac{2}{\beta - 3}} + \frac{\beta}{(3-\beta)^2} \, \frac{1}{t^2}$$

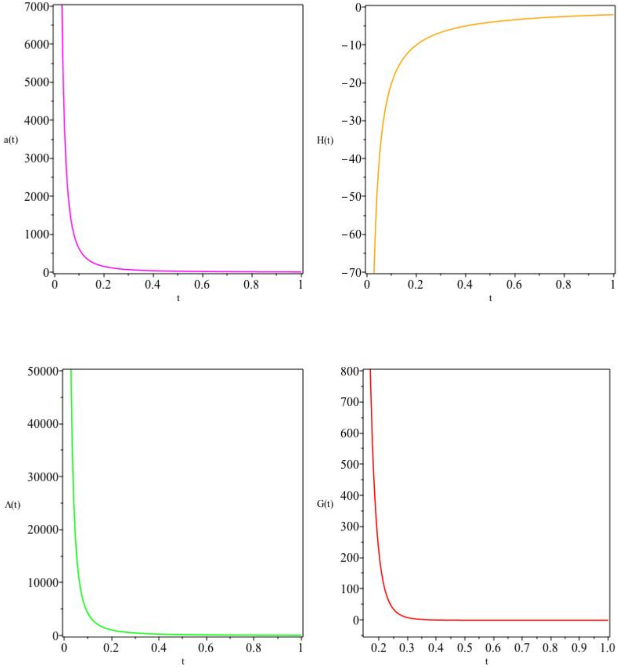
$$G = -\int \frac{\dot{\Lambda}}{8\pi\rho} dt = G_0 - \frac{c^2(\beta - 3)^4}{8\pi\rho_0(\beta - 2)^3} \left[C(\beta - 3) t^{\frac{2\beta}{3-\beta}} - \frac{\alpha(\beta - 2)}{2} t^{\frac{4}{3-\beta}} \right]$$

and the metric variables can be computed as

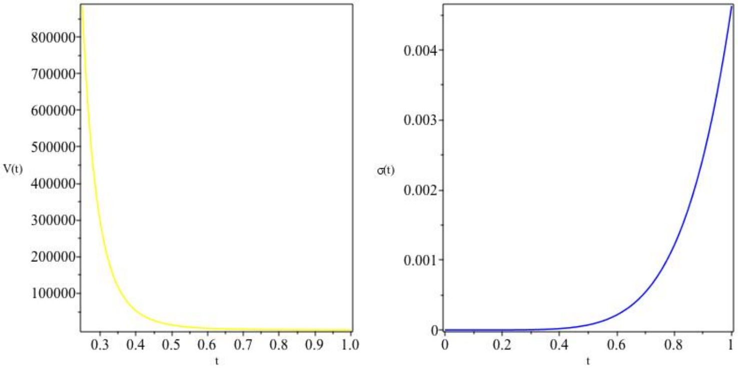
$$A = \left[(\beta - 3) \sqrt{\frac{c}{\beta - 2}} \;\; t \right]^{\frac{1}{3-\beta}}$$

$$B = B_0 \left[(\beta - 3) \sqrt{\frac{c}{\beta - 2}} \;\; t \right]^{\frac{1}{3-\beta}} \times \exp\left\{ \frac{k(\beta - 3)}{3} \left[(\beta - 3) \sqrt{\frac{c}{\beta - 2}} \right]^{\frac{3}{\beta - 3}} t^{\frac{\beta}{3-\beta}} \right\}$$

$$C = C_0 \left[(\beta - 3) \sqrt{\frac{c}{\beta - 2}} \;\; t \right]^{\frac{1}{3-\beta}} \times \exp\left\{ -\frac{k(\beta - 3)}{3} \left[(\beta - 3) \sqrt{\frac{c}{\beta - 2}} \right]^{\frac{3}{\beta - 3}} t^{\frac{\beta}{3-\beta}} \right\}$$



a(t)
7000
6000
5000
4000
3000
2000
1000
0
0
0.2
0.4
0.6
0.8
1
t
H(t)
0
−10
−20
−30
−40
−50
−60
−70
0
0.2
0.4
0.6
0.8
1
t
A(t)
50000
40000
30000
20000
10000
0
0
0.2
0.4
0.6
0.8
1
t
G(t)
800
700
600
500
400
300
200
100
0
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
t



V(t)
800000
700000
600000
500000
400000
300000
200000
100000
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
t
σ(t)
0.004
0.003
0.002
0.001
0
0
0.2
0.4
0.6
0.8
1
t

- The above results show that the model has a point of initial singularity (Big Bang) at $t = 0$
- The model represents a contracting universe solution where V and both Λ and G simultaneously decrease, while H, θ, σ and ρ all increase as time increases, with H asymptotically approaching zero from below
- Note that the model produces a constant deceleration of the expansion for acceptable values of β
- The expansion rate θ of the universe slows down as time increases, with the expansion eventually stopping, the ratio of $|\sigma/\theta| \to \infty$ as $t \to \infty$ thus predicting that the universe in this model becomes more anisotropic at late times
- Moreover, as $t \to \infty$, the metric components A, B and C approach zero rapidly, and $a \to 0$, which, with an even more rapidly increasing energy density, potentially results in a Big Crunch

Numerical Solutions

Here we use numerical integration methods and solve the coupled system:

$$
\begin{aligned}
& 8\pi\rho\dot{G} + \dot{\Lambda} = 0 \\
& \frac{\ddot{a}}{a} + (2-\beta)\frac{\dot{a}^2}{a^2} - \frac{2+\alpha}{a^2} = 4\pi G(t)\rho(1-w)
\end{aligned}
$$

for realistic perfect fluids dust matter, radiation and dark energy).
Rewerite system as:

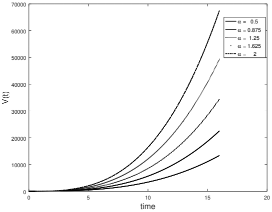
$$
\begin{aligned}
\frac{da}{dt} &= Z \\
\frac{dZ}{dt} &= -\frac{a_1\ Z^2}{a} + \frac{b_1}{a} + \frac{c_1 G}{a^{3w+2}} \\
\frac{dG}{dt} &= a_2\ Z\ a^{3w} + b_2\ Z^3\ a^{3w} - c_2\ \frac{Z}{a}\ G
\end{aligned}
$$

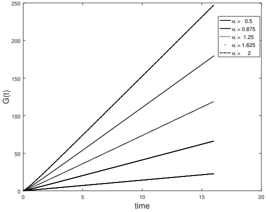
where $a_1 = (2-\beta)$, $b_1 = (2+\alpha)$, $c_1 = 4(1-w)\rho_0$,
$a_2 = (\alpha(1-\beta) - 2\beta)/4\rho_0$, $b_2 = \beta(3-\beta)/4\rho_0$, and $c_2 = \beta\ (1-w)$

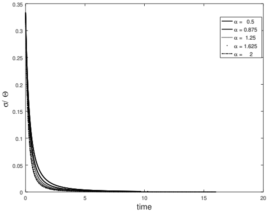
- It is important to know that the average scale factor of Bianchi type-V model $a(t) \neq 0$ as $t \to 0$ which implies $\dot{a} \neq 0$, $\dot{Z} \neq 0$ and $\dot{G} \neq 0$ are finite at the time of the Big Bang

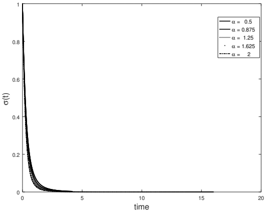
- System numerically solved for non-stiff perfect fluid models
- First attempt at numerical results for $\beta = 0.5$ and several values of α in the range $1 \leq \alpha \leq 2$
- Preliminary observations:
  - In each model, V and G increase with time except G in the dark-energy-dominated model, which increases at early times and then decreases later on
  - σ, σ/θ and Λ decrease with time, which indicates that the universe in each model becomes isotropic at late times
  - The deceleration parameter q behaves differently depending on the EoS of each model, that is, for a matter-dominated universe where $w = 0$, it asymptotically increases from a negative value to reach a constant value at zero and for a radiation-dominated universe, it asymptotically increases from a negative to positive value less than 1 as α increases, and for a dark-energy-dominated universe when EoS $w = -1$, q asymptotically negatively increases to approach a negative value less than -1

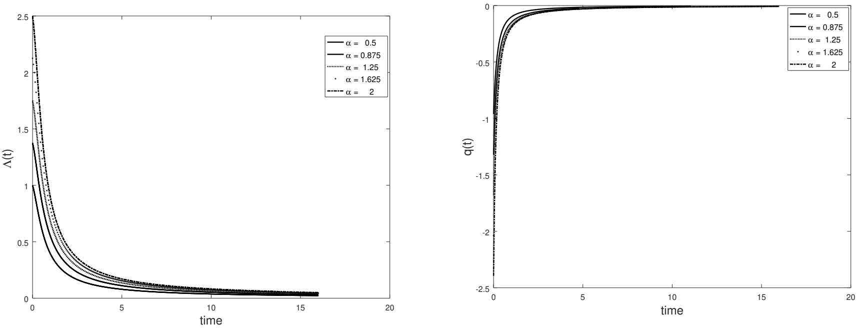
The Case of Dust





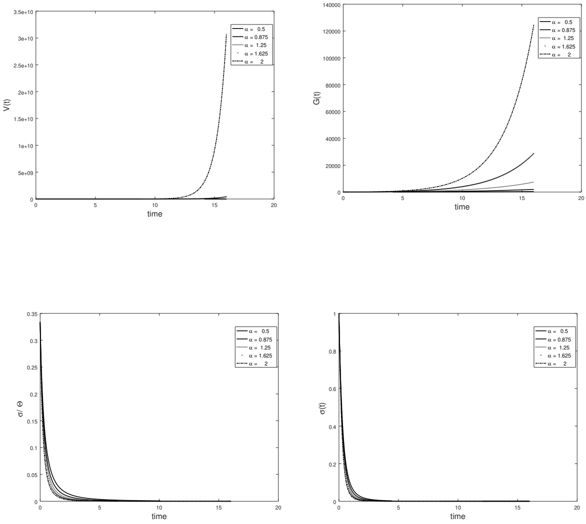


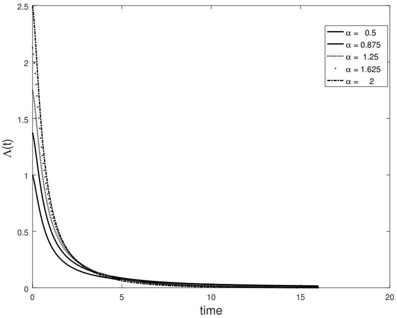




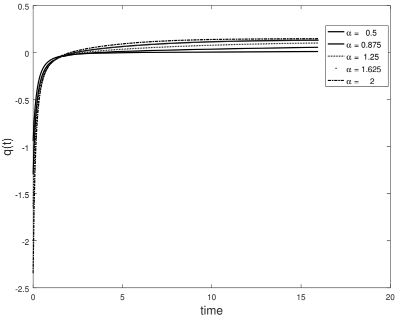
Λ(t)
time
α = 0.5
α = 0.875
α = 1.25
α = 1.625
α = 2
q(t)
time
α = 0.5
α = 0.875
α = 1.25
α = 1.625
α = 2

The Case of Radiation



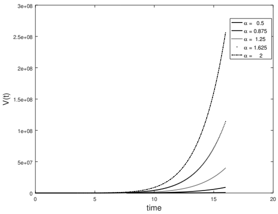


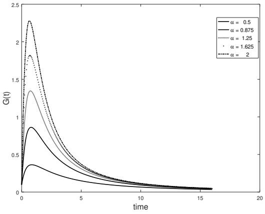
Λ(t)
time
α = 0.5
α = 0.875
α = 1.25
α = 1.625
α = 2

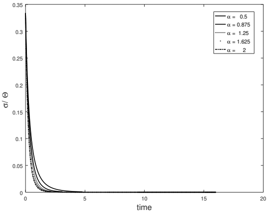


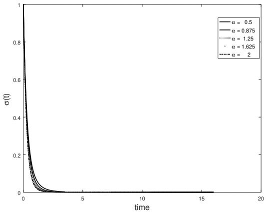
q(t)
time
α = 0.5
α = 0.875
α = 1.25
α = 1.625
α = 2

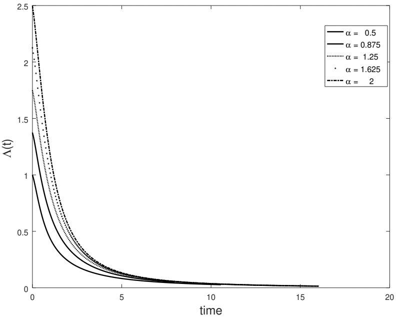
The Case of Dark Energy

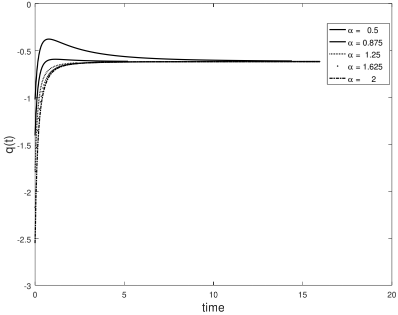









α = 0.5
α = 0.875
α = 1.25
α = 1.625
α = 2
Λ(t)
time
0
0.5
1
1.5
2
2.5
5
10
15
20


α = 0.5
α = 0.875
α = 1.25
α = 1.625
α = 2
q(t)
time
0
-0.5
-1
-1.5
-2
-2.5
-3
5
10
15
20

Summary

- General exact solutions for Bianchi Type-V cosmological models for a stiff perfect fluid with time varying cosmological and gravitational 'constants' without prior choice of the quantum-field-theoretically adjustable parameters α and β that define the cosmological constant as $\Lambda = \alpha/a^2 + \beta H^2$
- Two cosmological models obtained with a choice of suitably fixed values of β and through the transformation of the generalized Friedman equation into a special case of the Emden-Fowler equation
- The dynamical and kinematical parameters of each model are computed exactly
    - While one of these models results in a universe that asymptotically isotropizes at late times, the other becomes increasingly anisotropic
- Simple numerical computation implemented to see the general behaviour of the cosmological parameters for more realistic perfect-fluid models; work in progress...
- As more precise data become available, it will, *in principle*, be possible to constrain the different model parameters that we chose arbitrarily in this study to get a better picture of this class of cosmological models